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An Elementary Local Trace Formula for real Symmetric Spaces

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1. Introduction

Suppose that G is a connected Lie group, and δ an automorphism of order 2 of G, i.e., δ is an involution of G. Let us denote by G^{δ} the subgroup of the fixed points of the action of δ on G, and by $G_{\circ}^{\delta}$ the identity connected component of G^{δ}. If H is a subgroup of G such that

$$G_{\circ}^{\delta} \subseteq H \subseteq G^{\delta},$$

the quotient space $H\backslash G$ is called a symmetric space. Thus a symmetric space is characterized by the data (G, H, δ). In this paper we assume that H is open (hence closed, being a topological group). The group G acts on $H\backslash G$ and one can ask for the existence of a G-invariant measure on $H\backslash G$. According to [Borb] we know that when the module function d_G of G coincides with the module function d_H on H, then such a measure exists. Indeed, we assume that this condition holds and a G-invariant measure is selected on the quotient $H\backslash G$. Let $L^2(H\backslash G)$ be the Hilbert space of square integrable measurable functions on $H\backslash G$, with respect to the chosen invariant measure. The Plancherel formula gives a decomposition of the regular representation of G on $L^2(H\backslash G)$ as a direct sum of a continuous and a discrete part. In particular, when K is the compact subgroup of G obtained as the fixed points of a Cartan involution θ that commutes with δ, then under the following rank condition $(*)$ the discrete spectrum is non-empty.

$$(*) \quad rank(H\backslash G) = rank(H \cap K\backslash K).$$

Actually, throughout this paper, we assume that the above rank condition holds.

Based on the above condition the main purpose of this paper is to give a trace formula for the restriction of the regular represntation on the discrete spectrum. The general method used in our work is based on the ideas of J. Arthur developed in his work on the local trace formula. Our formula is not as complete as the local trace formula of J. Arthur. Nevertheless, since a group is itself a symmetric space, we can claim that when the symmetric space is attached to a connected reductive algebraic group over the reals then the trace formula of the present article is a generalization of the local trace formula of a connected real reductive algebraic group. The trace

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formula in our work is in essence an analytical trace formula. It is an identity which consists of two expressions, the geometric expansion, and the spectral expansion. In calculating the spectral expansion we have used the Plancherel formula for H, and in calculating the geometric expansion we have used a version of the Weyl integration formula for H.

2. A Review of Symmetric Spaces

Suppose that G is a connected reductive Lie group and δ an involution of G. Let G^δ be the subgroup of the fixed points of the action of δ on G, and $G^\delta_\circ$ the identity connected component of the closed subgroup G^δ. Let H be a subgroup of G such that:

$$G^\delta_\circ \subseteq H \subseteq G^\delta.$$

Then the quotient space $H\backslash G$ is called a *symmetric space* with the *isotropy group* H. Here we assume that the isotropy subgroup of the symmetric space is both open and connected. Depending on the structure of G, and H, the symmetric space $H\backslash G$ carries the structure of a Riemannian space.

It is known that there is a Cartan involution θ of G which commutes with the involution δ (see the work of Berger [Berg]). Assume that K is the maximal compact subgroup of G defined by the fixed points set of θ. One is interested in the Lie algebra decomposition of G according to the ± 1 eigenspaces of θ and δ. More precisely, let us denote the derivative of both θ and δ by the same letters. Let $\mathfrak{h}$ (resp. $\mathfrak{k}$) be the $+1$ eigenspace of δ (resp. θ), and $\mathfrak{q}$ (resp. $\mathfrak{p}$) be the -1 eigenspace of δ (resp. θ) on the Lie algebra $\mathfrak{g} = LieG$. Then

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-,$$

where

$$\mathfrak{g}_+ = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q},$$

$$\mathfrak{g}_- = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}.$$

Observe that $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are respectively the $+1$ and -1 eigenspace of $\delta\theta$ on $\underline{\mathfrak{g}}$. Let B be the non-degenerate G-invariant bilinear form on $\mathfrak{g}$, i.e., the *Killing form.* Then, it follows that all of the decompositions above are orthogonal direct sum decompositions with respect to the inner product

$$\langle X, Y\rangle_\theta = -B(X, \theta Y).$$

The *rank* of symmetric space $H\backslash G$, is by definition the dimension of a maximal abelian subspace in $\mathfrak{q}$. Observe that if $K_H = K \cap H$, then the quotient space $K_H\backslash K$

is a symmetric space associated to K with the involution δ. In fact $K^\delta = K \cap H$. Flensted-Jensen [F-J] has shown that if the following rank condition

$$rank(H\backslash G) = rank(K_H\backslash K), \tag{1}$$

holds, then the regular representation of G on the Hilbert space $L^2(H\backslash G)$ has a discrete spectrum. Thus, it is fundamental to know the decomposition of the regular rpresentation. let us first give an example. Suppose that $H\backslash G$ is a compact symmetric space. Then, by a result of Mostow [Mos], it is known that G must be compact. This implies in particular that the only part of the spectral decomposition of the regular representation is discrete.

In general, however, the problem of the spectral decomposition of the regular representation is considerably more difficult. For example, for a connected Lie group as a symmetric space, the problem of the spectral decomposition is solved by Harish-Chandra using the Plancherel formula. On the other hand, for a symmetric space $H\backslash G$ the problem is solved in the works [Ban], [BanSch], and [BanSch1].

Let us fix a maximal abelian subspace $\mathfrak{a}_\mathfrak{q}$ of $\mathbf{p} \cap \mathfrak{q}$. Let $\Delta^+ = \Delta^+(\mathfrak{a}_\mathfrak{q}, \mathfrak{g}_+)$ be a positive system of restricted roots for $\mathfrak{a}_\mathfrak{q}$ in $\mathfrak{g}_+$ [Hel]. Let $A_\mathfrak{q} = exp(\mathfrak{a}_\mathfrak{q})$, and

$$\mathfrak{a}_\mathfrak{q}^+ = \{T \in \mathfrak{a}_\mathfrak{q} : \langle \alpha, T\rangle_\theta \geq 0 \ \ \forall \alpha \in \Delta^+\}.$$

Then, the following Cartan decomposition holds:

$$G = HA_\mathfrak{q}^+K, \tag{2}$$

where $A_\mathfrak{q}^+ = exp(\mathfrak{a}_\mathfrak{q}^+)$. For a proof of this Cartan decomposition which is based on the geometry of the symmetric space $H\backslash G$, we refer the reader to [F-J1]. Moreover, our assumption that H is connected implies that the middle part of the Cartan decomposition of an element of G is uniquely determined [Ban1].

Rather than working with a Levi component, we fix a split component by which the parabolic subgroups will be determined. More precisely, let us denote by $\mathcal{P}$ the set of all $\delta\theta$-stable parabolic subgroups of G containing $A_\mathfrak{q}$. It is well known that this set is finite. Given $P \in \mathcal{P}$, let us write

$$P = M_P A_P N_P,$$

for the corresponding Langlands decomposition. Then, by our assumption, $A_P \subseteq A_\mathfrak{q}$. Let us denote by $\mathcal{P}(A_\mathfrak{q})$ the possible non-empty subset of $\mathcal{P}$ such that $A_P = A_\mathfrak{q}$.

Since $G = PK$, an element $x \in G$ can be decomposed as the product

$$x = m_P(x) a_P(x) n_P(x) k_P(x).$$

Let $H_P(x)$ be the unique vector in $\mathfrak{a}_P = LieA_P$ such that

$$a_P(x) = exp(H_P(x)).$$

Then we can write

$$x = m_P(x)exp(H_P(x))n_P(x)k_P(x).$$

In particular, when $P \in \mathcal{P}(A_\mathfrak{q})$, denote the element $H_P(x)$ by $H_P^\mathfrak{q}(x)$. Let us denote by Δ_P the set of simple roots of $(P, A_\mathfrak{q})$.

3. The Expansions of the Kernel

With the above notations we define the regular representation by:

$$(R(y)\varphi)(x) = \varphi(xy) \quad x, y \in G, \ \ \varphi \in L^2.$$

In this paper, a test function on G is a compactly supported continuous function on G. We denote the space of these functions by $\mathcal{H}$.

Let us denote by $R(f)$ the extension of $R(y)$ to $\mathcal{H}$. Then $R(f)$ acts on L^2 by:

$$(R(f)\varphi)(x) = \int_G f(y)\varphi(xy)dy, \quad f \in \mathcal{H}. \tag{3}$$

After a change of variable we can write

$$(R(f)\varphi)(x) = \int_G f(x^{-1}y)\varphi(y)dy.$$

Using Fubini's theorem we can write:

$$(R(f)\varphi)(x) = \int_{H\backslash G}(\int_H f(x^{-1}ty)\varphi(ty)dt)dy,$$

or equivalently

$$(R(f)\varphi)(x) = \int_{H\backslash G}(\int_H f(x^{-1}ty)dt)\varphi(y)dy.$$

Thus, we have shown that

$$(R(f)\varphi)(x) = \int_{H\backslash G}(\int_H f(x^{-1}ty)dt)\varphi(y)dy. \tag{4}$$

From this we conclude that the kernel is:

$$K(x, y) = \int_H f(x^{-1}ty)dt, \quad t \in H, \ \ x, y \in G. \tag{5}$$

Let H_{reg} be the set of all regular elements of H. An element $\gamma \in H$ is called *elliptic* if its centralizer in H is compact modulo A_H°, the identity connected component of the split component A_H of H. For a regular elliptic element γ of H, let us denote by $\{\gamma\}$ the conjugacy class in H of γ. The set of all conjugacy classes defined in this way is denoted by $E(H)$. We then write:

$$E_{reg}(H) = E(H) \cap H_{reg}.$$

Let us recall that a maximal torus T of H is *elliptic* if T/A_H° is compact.Since the following discussion depends on a fixed measure, it is important to explain how the measure is fixed. One begins by fixing a measure on A_H°. This measure induces a measure on A_H°, and this in turn determines a canonical measure $d\gamma$ on $E_{reg}(H)$, which vanishes on the complement of H_{reg} in E_{reg}, and such that:

$$\int_{E_{reg}(H)} \eta(\gamma)d\gamma = \sum_{\{T\}} |W(H,T)|^{-1} \int_T \eta(t)dt,$$

for any continuous function η of compact support on $E_{reg}(H)$. Here $\{T\}$ is a set of representatives of H-conjugacy classes of maximal tori H with elliptic T. Moreover, $W(H,T)$ is the Weyl group of (H,T), and dt is the Haar measure on the compact group T/A_H° (cf. [A, page 16]).

Suppose that $M_\circ^H$ is a fixed minimal Levi component of H. Let $\mathcal{L}^H$ be the set of all Levi components of H that contains $M_\circ^H$. Let, W^H be the Weyl group of the pair (H, A_H°). If M_H is a Levi component of H, denote by W^{M_H} the Weyl group of the pair $(M_H, A_{M_H}^\circ)$.

Lemma 1. Let $D(\gamma)$ be the Weyl discriminant of γ, and for each $x \in G$, let $g_x(t) = f(x^{-1}tx)$, with $t \in H$. Then

$$K(x,x) = \sum_{M_H \in \mathcal{L}^H} |W| \int_{E_{reg}(H)} |D(\gamma)| \left(\int_{A_{M_H}^\circ \backslash H} g_x(x_1^{-1}\gamma x_1)dx_1 \right) d\gamma,$$

where $|W| = |W^{M_H}||W^H|^{-1}$, and $E_{reg}(M_H)$ is the same as $E_{reg}(H)$ but with H replaced by M_H.

Proof. For a function $h' \in C_c^\infty(H)$ we have

$$\int_H h'(u)du = \sum_{M_H \in \mathcal{L}^H} |W| \int_{E_{reg}(M_H)} |D(\gamma)| \left(\int_{A_H^\circ \backslash H} h'(u^{-1}\gamma u)du \right) d\gamma.$$

Now, note that $g_x \in C_c^\infty(H)$. Thus, the identity can be applied to g_x in place of h'. On the other hand, since

$$K(x,x) = \int_H g_x(\gamma)d\gamma,$$

the result follows. □

The expression given by the lemma is the first geometric expansion of the kernel. We now determine the first spectral expansion of the kernel. To this end, we need to define the function h_x, for each $x \in G$, by:

$$h_x(v) = \int_H f(x^{-1}vux)du, \quad u, v \in H.$$

Then $h_x \in C_c^\infty(H)$, and

$$h_x(1) = K(x, x).$$

The Plancherel formula gives an expression for $h_x(1)$ based on the character of an induced representation. We do not recall the notaions here, but we refer the reader to the work of J. Arthur [A] for all undefined notations. The result is:

$$h_x(1) = \sum_{M_H \in \mathcal{L}^H} |W| \int_{\Pi_2(M_H)} m(\sigma) tr(I_{P_H}(\sigma, h_x)) d\sigma. \tag{6}$$

As a consequence we have proved the following result, the first spectral expansion of the kernel.

Lemma 2. We have:

$$K(x, x) = \sum_{M_H \in \mathcal{L}^H} |W| \int_{\Pi_2(M_H)} m(\sigma) tr(I_{P_H}(\sigma, h_x)) d\sigma,$$

where P_H denotes a parabolic subgroup of H with the Levi component M_H. □

4. The Truncation

We first need to define the notion of orthogonal set, in a way suitable for our purposes.

Let $P, P' \in \mathcal{P}(A_\mathfrak{q})$. Say that P, P' are *adjacent* if their chambers have an hyperplane in common. With respect to Δ_P a finite set of points Y_P in $\mathfrak{a}_\mathfrak{q}$, indexed by $P \in \mathcal{P}(A_\mathfrak{q})$ is called $A_\mathfrak{q}$*-orthogonal* (or, *orthogonal*) if for any two adjacent groups P, P' whose chambers in $\mathfrak{a}_\mathfrak{q}$ share the wall determined by the simple root α in $\Delta_P \cap (-\Delta_{P'})$ satisfies:

$$Y_P - Y_{P'} = r\alpha^\vee,$$

for a real number $r = r(P, P')$. When $r > 0$, the set is called *positive* $A_\mathfrak{q}$ *-orthogonal.* Note that $\alpha^\vee$ is the co-root associated to the simple root $\alpha \in \Delta_P$ (cf. [A2]).

Suppose that T is any point in $\mathfrak{a}_\mathfrak{q}$. Denote by $\mathcal{P}_\circ(A_\mathfrak{q})$ the set of minimal elements of $\mathcal{P}(A_\mathfrak{q})$. Let $T_\circ$ be the unique translate of T under the action of the Weyl group of $(G, A_\mathfrak{q})$. Then the set

$$\overline{T} = \{T_\circ : P_\circ \in \mathcal{P}_\circ(A_\mathfrak{q})\}$$

is a positive orthogonal set. We assume that $\overline{T}$ is highly regular in the sense that its distance from any singular hyperplane in $\mathfrak{a}_{\mathfrak{q}}$ is large. Let us write $S_{\mathfrak{q}}(T)$ for the convex hull in $\mathfrak{a}_{\mathfrak{q}}/\mathfrak{a}_G$ of the orthogonal set $\overline{T}$ (note that $\mathfrak{a}_G = LieG = \mathfrak{g}$). The truncation method is originally based on the properties of Langlands' Combinatorial Lemma and its role in the theory of the Eisenstein series [Lan], [A1], and [Sho1]. The truncation is a process that trnsforms slowly increasing functions to rapidly decreasing functions. By this method in the trace formulae one determines a truncated kernel that has a convergent integral. This has been first applied in the global trace formula (see [Sho] for more informations in the global non-twisted case, and [Sho1] for the local twisted case). The similar method can be used for the same purpose in the study of the local trace formula in the non-twisted case [A], and here for the local trace formula attached to the symmetric spaces.

Let A_G° be the identity connected component of A_G. Now, let $x \in G$ be given, then by the Cartan decomposition of $H\backslash G$ we can write

$$x = h(x)a(x)k(x), \;\; h(x) \in H, \;\; a(x) \in A_G^\circ\backslash G, \;\; k(x) \in K.$$

We now fix a highly regular point $T \in \mathfrak{a}_{\mathfrak{q}}$. Let $a(x) \in A_G^\circ\backslash G$ be the middle component of the Cartan decomposition. Let $u(x,T)$ be the characteristic function of the set

$$\mathcal{U} = \{x \in A_G^\circ\backslash G : H_P(a(x)) \in S_{\mathfrak{q}}(T)\}.$$

Note that $H_P(a(x))$ lies in $logA_{\mathfrak{q}}$. The function $u(x,T)$ is called the *truncation function.* This function is applied to the kernel $K(x,x)$ to yield the *truncated kernel*

$$K^T(f) = \int_{A_G^\circ\backslash G} K(x,x)u(x,T)\,dx.$$

We can now prove:

Lemma 3. The integral defining $K^T(f)$ converges.

Proof. Note that $S_{\mathfrak{q}}(T)$ is a (large) compact subset of $\mathfrak{a}_{\mathfrak{q}}/\mathfrak{a}_G$, hence $u(x,T)$ is the characteristic function of a (large) compact subset of $A_G^\circ\backslash G$. □

Lemma 4. with the notations as above we have the following geometric expression for the kernel $K^T(f)$:

$$K^T(f) = \sum_{M_H \in \mathcal{L}^H} |W| \int_{E_{reg}(M_H)} K^T(\gamma, g)\,d\gamma,$$

where $K^T(\gamma,g)$ is given by:

$$|D(\gamma)| \int_{A_G^\circ\backslash G} \int_{A_{M_H}^\circ\backslash H} g_x(x_1^{-1}\gamma x_1)u(x,T)\,dx_1 dx, \qquad (7)$$

in which $x \in G$, $x_1 \in H$.

Proof. Substituting the expression for $K(x,x)$ given in Lemma 1, in the definition $K^T(f)$ and then removing the finite summation over $\mathcal{L}^H$, the result follows. □

Similarly, one can prove the following result.

Lemma 5. Let the notations be as above, then the following spectral expansion for $K^T(f)$ holds:

$$\sum_{M_H \in \mathcal{L}^H} |W| \int_{\Pi_2(M_H)} K^T(\sigma, h_x)\, d\sigma,$$

where, $K^T(\sigma, h_x)$ is given by:

$$m(\sigma) \int_{A_G^\circ \backslash G} tr(I_{P_H}(\sigma, h_x))u(x,T)\, dx.$$

□

We still have to modify the expansions above. To do so, let

$$u_{\mathfrak{q}}(x_1, x_2, T) = \int_{A_G^\circ \backslash A_{\mathfrak{q}}^\circ} u(x_1^{-1} a x_2, T)\, da, \tag{8}$$

where, $x_1 \in H$, $x_2 \in G$, $a \in A_{\mathfrak{q}}^\circ$.

Lemma 6. The following expression for $K^T(\gamma, g)$ holds:

$$|D(\gamma)| \int_{A_{\mathfrak{q}}^\circ \backslash G} \int_{A_{M_H}^\circ \backslash H} g_{x_2}(\gamma) u_{\mathfrak{q}}(x_1, x_2, T)\, dx_1 dx_2.$$

Proof. In (7) one can decompose the integral over $A_G^\circ \backslash G$ in two integrals, one over $A_{\mathfrak{q}}^\circ \backslash G$, and the other over $A_G^\circ \backslash A_{\mathfrak{q}}^\circ$. Then, change the variable by setting $x_2 = x_1 x$. from this it follows that

$$x_1^{-1} x_2 = x, \quad dx = dx_2.$$

hence we have that $K^T(\gamma, g)$ equals:

$$|D(\gamma)| \int_{A_{\mathfrak{q}}^\circ \backslash G} \int_{A_{M_H}^\circ \backslash H} g_{x_2}(\gamma) u(x_1^{-1} x_2, T)\, dx_1 dx_2.$$

To complete the proof decompose the first integral which is with respect to dx_2, in two integrals mentioned above. This causes the change of $x_1^{-1} x_2$ to $x_1^{-1} a x_2$. □

5. Some Geometric Preparations

The central point in the geometric study of the trace formula in this section is the main geometric lemma, which according to it, the truncated kernel $K^T(\gamma, f)$

can be approximated by a weighted orbital integral. For establishing this lemma, we first need to define an appropriate orbital integral. This is acheieved by first introducing an $A_{\mathfrak{q}}$-orthogonal set whose associated weight factor will be used in the definition of the integral. For $P \in \mathcal{P}(\mathcal{A}_{\mathfrak{q}})$, $x_1 \in H$, and $x_2 \in G$, define

$$Y_P(x_1, x_2, T) = T + H_P(x_1) - H_{\overline{P}}(x_2),$$

where, $T \in \mathfrak{a}_{\mathfrak{q}}$, and $\overline{P}$ is the opposite parabolic subgroup of P. Then the points

$$\mathcal{Y} = \mathcal{Y}_{\mathfrak{q}}(x_1, x_2, T) = \{Y_P(x_1, x_2, T) : P \in \mathcal{P}(A_{\mathfrak{q}})\}, \tag{9}$$

form an $A_{\mathfrak{q}}$-orthogonal set. Let as before (cf. Section 4) $T_\circ$ be the translate of T under the Weyl group, and

$$d(T) = inf\{\alpha(T_\circ) : \alpha \in \Delta_P, P \in \mathcal{P}(A_{\mathfrak{q}})\}.$$

Then it can be shown that $\mathcal{Y}$ is positive if $d(T)$ is large with respect to x_1 and x_2 [A page 30]. To this orthogonal set one associates a weight factor in the same way discussed in [A page 30], but it will be parameterized by parabolic subgroups instead of the Levi components. In another word, we have to begin with the function $\sigma_{\mathfrak{q}} = \sigma_{A_{\mathfrak{q}}}$ which is the analogue of the function σ_M of [A (3.8)]. The function $\sigma_{\mathfrak{q}}$ is defined by:

$$\sigma_{\mathfrak{q}}(X, \mathcal{Y}) = \sum_{P \in \mathcal{P}(A_{\mathfrak{q}})} (-1)^{|\hat{\Delta}_P|} \varphi_P^{\Lambda}(X - Y_P),$$

where, $X \in \mathfrak{a}_{\mathfrak{q}}/\mathfrak{a}_G$, and the undefined notations are to be found in [A page 22]. Now, let us define the weight factor

$$v_{\mathfrak{q}}(x_1, x_2, T) = \int_{A_G^\circ \backslash A_{\mathfrak{q}}^\circ} \sigma_{\mathfrak{q}}(H_P(a), \mathcal{Y}(x_1, x_2, T))\, da. \tag{10}$$

Using this weight factor the following weighted orbital integral is defined:

$$J^T(\gamma, g) = |D(\gamma)| \int_{A_{\mathfrak{q}}^\circ \backslash G} g_{x_2}(\gamma) v_{\mathfrak{q}}(x_1, x_2, T)\, dx_1 dx_2.$$

As we have already mentioned for the purpose of the trace formula we need to compare the kernel $K^T(\gamma, g)$ with the orbital integral $J^T(\gamma, g)$. This is essentially done by the method of approximation. For this method a key is the notion of the norm, or distance function on the group and the associated symmetric space. Actually, it turns out that the norm is the same as the height function.

Let $\Lambda : G \longrightarrow GL(V)$ be a finite dimensional representation of G, and $\{v_1, \ldots, v_n\}$ a basis of V. The *height* of any vector

$$v = \sum_{i=1}^{n} \lambda_i v_i, \quad \lambda_i \in I\!R,$$

is the Euclidean norm $\| \ \|'$ of v, i.e.,

$$\|v\|' = (\sum_{i=1}^{n} |\lambda_i|^2)^{\frac{1}{2}}.$$

Then, the *height function* (or *height*) of an element $g \in G$ is defined by:

$$\|g\|' = \|\Lambda(g)\|'.$$

One can define an height function on G that comes from the inner product of $\mathfrak{g} = LieG$. Indeed, we can equip $\mathfrak{g}$ with the structure of a real Hilbert space by defining the inner product $\langle X, Y\rangle_\theta$, and then the norm

$$\|X\|_\theta = \langle X, X\rangle_\theta^{\frac{1}{2}}, \quad X \in \mathfrak{g}.$$

One can also extend this norm to G by defining the Hilbert-Schmidt inner product of $Ad(x)$, $Ad(y)$, with $x, y \in G$ as follows:

$$(x, y) = dim(\mathfrak{g})^{-1} tr_{\mathfrak{g}}(Ad(y)^* Ad(x)),$$

where the adjoint operator $Ad(y)^*$ of $Ad(y)$ is defined with respect to $\langle , \rangle_\theta$ and given by $Ad(\theta(y)^{-1})$. Thus we can write

$$(x, y) = dim(\mathfrak{g}^{-1}) tr_{\mathfrak{g}}(Ad(x\theta(y^{-1})).$$

The norm of $x \in G$ is then defined by:

$$\|x\| = (x, x)^{\frac{1}{2}},$$

and it satisfies the following basic properties:
(H1) $\|xy\| \leq (dim\mathfrak{g})^{-1}\|x\| \, \|y\|$,
(H2) $\|x\| = \|x^{-1}\| = \|\theta(x)\| \geq 1$,
(H3) $\|k_1 x k_2\| = \|x\|, \quad k_1, k_2 \in K$
(H4) There are constants $c_1, c_2 > 0$ such that if $x = expX$, with $X \in \mathfrak{p}$, then

$$exp(c_1\|X\|) \leq \|X\| \leq exp(c_2\|X\|),$$

where $\|X\|$ denotes the norm $\|X\|_\theta$.
(H5) $\|a\| \leq \|an\|, \quad a \in A, n \in N$.
For a proof of these facts see for example [BanSch page 112]. In paricular, the following result shows that for our height function we can take the above norm on G.

Lemma 7. The norm $\|.\|$ on G satisfies all the properties of the height function.

Proof. We have only to show that $\|xy\| \leq \|x\|\|y\|$. But, this is done by the normalization which transforms the property (H1) to the sought inequality [A page 26]. $\square$

As we will see the approximation of the kernel $K^T(f)$ by the weighted orbital integral $J^T(\gamma, f)$ is based on the study of the difference

$$u_{\mathfrak{q}}(x_1, x_2, T) - v_{\mathfrak{q}}(x_1, x_2, T).$$

The approximation of this difference is investigated in the following theorem.

Theorem 1. Let $\beta > 0$ be a fixed positive number. Suppose that ϵ_2 is a positive small number so that from

$$\|x_i\| \leq exp(\epsilon_2\|T\|) \quad i = 1, 2. \tag{11}$$

follows that

$$|u_{\mathfrak{q}}(x_1, x_2, T) - v_{\mathfrak{q}}(x_1, x_2, T)| \leq Cexp(-\epsilon_1\|T\|), \tag{12}$$

for some positive constants C, ϵ_1. Then the constants C, ϵ_1, ϵ_2 can be chosen such that (12) holds for all T with $d(T) \geq \beta\|T\|$, and all x_1, x_2 in the set

$$\{x \in G : \|x\| \leq exp(\epsilon_2\|T\|)\}. \tag{13}$$

This theorem is fundamental for the geometric theory of the trace formula and we devote a major part of the next section to its proof.

6. Proof of Theorem 1, and the Geometric Expansion

We begin this section by a lemma that generalizes Lemma 5.2 of [A] to a symmetric space.

Lemma 8. Suppose that X, X' are two points in $\mathfrak{g}$ and Y, Y_1 are two points in $\mathfrak{a}_{\mathfrak{q}}^+$ such that

$$exp(X)^{-1}exp(Y)exp(X') = h(exp(Y_1))k,$$

where $h \in H, k \in K$. Then,

$$\|Y_1 - Y\|' \leq \|X\|' + \|X'\|'.$$

Proof. When the isotropy group in the symmetric space is compact, the Lemma is proved in [A Lemma 5.2]. We thus assume that the isotropy group in the symmetric space is non-compact, and also the symmetric space is non-Riemannian. Let $\mathfrak{a}_\circ$ be the extension of $\mathfrak{a}_\mathfrak{q}$ to a maximal abelian subspace of $\mathfrak{p}$. Let $A_\circ = exp(\mathfrak{a}_\circ^+)$. Since $A_\circ$ is maximal the following Cartan decomposition also holds:

$$G = KA_\circ K.$$

In fact, according to [H-C page 243], if ℓ is the rank of the symmetric space G/K then there is a connected abelian Lie subgroup A_ℓ of G such that

$$G = KA_\ell K.$$

Now, since Y, Y_1 are elements of $\mathfrak{a}_\mathfrak{q}^+$, they also belong to $\mathfrak{a}_\circ^+$. Hence, we have to show that there are elements $k_1, k_2 \in K$ so that from

$$exp(X)^{-1}exp(Y)exp(X') = h(exp(Y_1))k$$

it follows that the left-hand side of the equality equals to

$$k_1 exp(Y_1)k_2,$$

for some $k_1, k_2 \in K$. But, this fact follows from the above Cartan decomposition for the symmetric space G/K. The lemma now follows from the proof of [A Lemma 5.2] in the Riemannian case. □

Lemma 9. Let $\mathcal{Y}$ be a $A_\mathfrak{q}$-orthogonal set. For a parabolic group $P \in \mathcal{P}(A_\mathfrak{q})$ one has:

$$S_\mathfrak{q}(\mathcal{Y}) \cap \mathfrak{a}_\mathfrak{q}^+ = \{X \in \mathfrak{a}_\mathfrak{q}^+ : \varpi(X - Y) \leq 0,\ \varpi \in \hat{\Delta}_P\},$$

where $Y \in S_\mathfrak{q}(\mathcal{Y})$.

Proof. The proof is exactly the same as the proof of Lemma 3.1 of [A]. In fact, the proof of our lemma can even be deduced directly from [A2 Lemma 3.2]. □

To prepare the ground for the proof of the Theorem, we have to invoke a result of Langlands which is based on his combinatorial lemma, and gives a description of the quotient space $A_\mathfrak{q}\backslash A_G$ similar to the reduction theory [Sho].

Let ϵ be a positive number which depends on G and the constant β of the Theorem. In another word $\epsilon = \epsilon(G, \beta)$. Let $Q \in \mathcal{P}$, and $A_\mathfrak{q}(Q, \epsilon)$ be the set

$$\{a \in A_\mathfrak{q}\backslash A_G : \sigma_P^Q(H_P(a), \epsilon T)\tau_Q(H_P(a) - \epsilon T_Q) = 1\},$$

with the notations of [A].

Proposition 1. (i) The weight factor $u_\mathfrak{q}(x_1, x_2, T)$ equals the sum over $Q \in \mathcal{P}$ of the integrals

$$\sum_{Q\in\mathcal{P}} \int_{A_\mathfrak{q}(Q,\epsilon)} u(x_1^{-1}ax_2, T)\, da.$$

(ii) The weight factor $v_{\mathfrak{q}}(x_1,x_2,T)$ equals the sum over $Q \in \mathcal{P}$ of the integrals

$$\sum_{Q\in\mathcal{P}} \int_{A_{\mathfrak{q}}(Q,\epsilon)} \sigma_P(H_P(a),\mathcal{Y}(x_1,x_2,T))\, da.$$

Proof. Let us consider the orthogonal set

$$Y(\epsilon) = \{\epsilon T_P : P \in \mathcal{P}(A_{\mathfrak{q}})\}.$$

According to the consequence of Langlands' combinatorial lemma, the orthogonal set $Y(\epsilon)$ satisfies the property:

$$\sum_{Q\in\mathcal{P}} \sigma_P^Q(X,Y(\epsilon))\tau_Q(X-\epsilon T_Q) = 1, \tag{14}$$

for any point $X \in \mathfrak{a}_{\mathfrak{q}}$. This shows that for a given Q the set $A_{\mathfrak{q}}(Q,\epsilon)$ consists of those points $X = H_P(a)$ that satisfy identity (14). Thus when Q varies over $\mathcal{P}$, we see that $u_{\mathfrak{q}}(x_1,x_2,T)$ and $v_{\mathfrak{q}}(x_1,x_2,T)$ which are given by the integrals over a in $A_{\mathfrak{q}}\backslash A_G$ of two compactly supported functions $u(x_1^{-1}ax_2,T)$ and $\sigma_P(H_P(a),\mathcal{Y}(x_1,x_2,T))$ respectively are decomposed in a finite sum over $\mathcal{P}$. □

Using the above result one can consider the following integrals

$$\int_{A_{\mathfrak{q}}(Q,\epsilon)} u(x_1^{-1}ax_2,T)\, da, \tag{15}$$

$$\int_{A_{\mathfrak{q}}(Q,\epsilon)} \sigma_P(H_P(a),\mathcal{Y}(x_1,x_2,T))\, da, \tag{16}$$

and try to study their difference. Nevertheless, it is enough to study the difference between their summands.

We now assume that Q is fixed as before, and we also fix an element $a \in A_{\mathfrak{q}}(Q,\epsilon)$. Let $P_\circ \in \mathcal{P}(A_{\mathfrak{q}})$ be a minimal parabolic subgroup such that $P_\circ \subset Q$. Denote by $\overline{Q}$ the opposite parabolic subgroup of Q, and let $H_\circ(x)$ be the vector $H_{P_\circ}(x)$. Since x_1 also belongs to G, there is an element $t = t_Q(x_1,x_2,a)$ in $A_{\mathfrak{q}}$ such that the product

$$m_Q(x_1)^{-1}am_{\overline{Q}}(x_2)$$

can be written as

$$k^{-1}tk', \quad k,k' \in K,$$

and $H_\circ(t) \in \mathfrak{a}_{\mathfrak{q}}^+$ (cf. [A page 36]).

Lemma 10. Suppose that x_1,x_2 satisfy (11). Then, $\mathcal{Y}_{\mathfrak{q}}(x_1,x_2,T)$ (cf. (9)) is a positive orthogonal set, and the characteristic function

$$\sigma_{\mathfrak{q}}(H_P(a),\mathcal{Y}_{\mathfrak{q}}(x_1,x_2,T)), \quad a \in A_{\mathfrak{q}}(Q,\epsilon)$$

equals 1 if and only if the vector

$$H_Q(t) = -H_Q(x_1) + H_Q(a) + H_{\overline{Q}}(x_2)$$

lies in the convex hull $S_{M_Q}(T)$.

Proof. This is exactly proved like Lemma 5.1 of [A]. □

To the elements k, k', t are associated two elements X, X' of the Lie algebra of $\mathfrak{g}$ as in [A pp 34-35]. Let

$$\zeta = exp(Ad(k)X), \quad \zeta' = exp(Ad(k')X').$$

Then, there is a point $t_1 = t_{1,Q}(x_1, x_2, a)$ in $A_{\mathfrak{q}}$ with the property that $H_\circ(t_1) \in \mathfrak{a}_{\mathfrak{q}}^+$ such that

$$\zeta^{-1} + \zeta' = k_1 t_1 k_1', \quad k_1, k_1' \in K.$$

At this stage, by Lemma 8, we see that

$$\|H_\circ(t_1) - H_\circ(t)\| \leq \|X\| + \|X'\|.$$

Then from (11), (12) we conclude that

$$\|H_\circ(t_1) - H_\circ(t)\| \leq Cexp(-\epsilon'\|T\|),$$

for a constant ϵ' which depends on $\beta, \epsilon_1, \epsilon_2$. Then the proof of the Theorem follows from the approximation methods of the volume of certain convex subsets described in [A pp 41-42]. □

Let us now return to the study of the geometric expansion. To begin, recall that $J^T(\gamma, f)$ is the weighted orbital integral attached to the weight factor $v_{\mathfrak{q}}(x_1, x_2, T)$ and given by:

$$|D(\gamma)| \int_{A_{\mathfrak{q}}^\circ \backslash G} \int_{A_{M_H}^\circ \backslash H} g_{x_2}(\gamma) v_{\mathfrak{q}}(x_1, x_2, T)\, dx_1 dx_2.$$

Let S be a fixed maximal torus in M_H which is compact modulo $A_{M_H}^\circ$, let $S_{reg} = S \cap H_{reg}$. Now, define

$$S(\epsilon, T) = \{x \in S_{reg} : |D(\gamma)| \leq exp(-\epsilon\|T\|)\}.$$

We would like to show that the kernel $K^T(f)$ can be estimated by

$$J^T(f) = \sum_{M_H \in \mathcal{L}^H} |W| \int_{E_{reg}(M_H)} J^T(\gamma, g)\, d\gamma. \tag{17}$$

The idea of the proof is not different from the proof of the similar result which leads to the geometric expansion of the local trace formula described in [A pp 30-34]. One shows that for any given $\epsilon > 0$, there is a constant c such that for any T,

$$\int_{S(\epsilon,T)} (|K^T(\gamma,g)| + |J^T(\gamma,g)|)\, d\gamma \leq c\, exp(-\epsilon\|T\|/2). \tag{18}$$

In fact, the product $x_1^{-1}ax_2$, with $x_1 \in H, x_2 \in G$, and $a \in A_{\mathfrak{q}}^\circ$ can be written as k_1bk_2, where, $k_1, k_2 \in K$ and $b \in A_\circ$ (note that $A_\circ$ is the abelian group introduced in the proof of Lemma 8), and by using the Cartan decomposition $KA_\circ K$. The rest of the proof then follows from the proof of Lemma 4.8 of [A].

The next step is to study the integral of the absolute value of the difference of $K^T(\gamma, f)$ and $J^T(\gamma, f)$ over the set $S - S(\epsilon, T)$. For this, the following result holds, and a proof for it is exactly the same as a proof in the local trace formula [A].

$$\int_{S-S(\epsilon,T)} |K^T(\gamma,g) - J^T(\gamma,g)|\, d\gamma \leq c_1 exp(-\epsilon_1\|T\|), \tag{19}$$

where the constant c_1 is given by

$$Cvol(A_{M_H}^\circ\backslash H) \int_{S_{reg}} |D(\gamma)| (\int_{A_G^\circ\backslash G} g_{x_2}(\gamma)\, dx_2)\, d\gamma.$$

From this it follows that for a constant C' and $\epsilon' > 0$

$$\int_{S_{reg}} |K^T(\gamma,g) - J^T(\gamma,g)|\, d\gamma \leq C' exp(-\epsilon'\|T\|), \tag{20}$$

for all T such that $d(T) \geq \beta\|T\|$. Hence, we have the following result.

Lemma 11. There are positive cnostants c'' and ϵ'' so that

$$|K^T(f) - J^T(f)| \leq C'' exp(-\epsilon''\|T\|)$$

for all T with $d(T) \geq \beta\|T\|$. □

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